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*Published in:*  
Computational Technologies in Concrete Structures

*Publication date:*  
2009

*Document Version*  
Publisher's PDF, also known as Version of record

[Link back to DTU Orbit](#)

*Citation (APA):*  
Larsen, K. P. (2009). Limit analysis of solid reinforced concrete structures. In *Computational Technologies in Concrete Structures* (pp. 154)

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# Limit Analysis of Solid Reinforced Concrete Structures

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## Abstract

Recent studies have shown that Semidefinite Programming (SDP) can be used effectively for limit analysis of isotropic cohesive-frictional continua using the classical Mohr-Coulomb yield criterion. In this paper we expand on this previous research by adding reinforcement to the model and a solid element for lower bound analysis of reinforced concrete structures is presented. The method defines the stress state at a point within the solid as a combination of concrete- and reinforcement stresses and yield criteria are applied to the stress components separately. This method allows for orthotropic reinforcement and it is therefore possible to analyze structures with complex reinforcement layouts. Tests are conducted to validate the method against well-known analytical solutions.

## 1 Introduction

The first reinforced concrete structures came about in the last part of the 18th century and concrete is today the most widespread construction material. In the last part of the 1970ties the first numerical solutions for the determination of the load bearing capacity of reinforced concrete structures was presented by (Fredsgaard & Kirk, 1979; Damkilde & Kirk, 1981). Since then, formulations for the load bearing capacity has been presented for a wide range of structural components: beams (Damkilde & Hoyer, 1993a), plates (Damkilde & Hoyer, 1993b), stringers (Damkilde *et al.*, 1994) and disks (Poulsen & Damkilde, 2000). Even though many different element types has been formulated, no real 3D modeling capabilities existed because the different elements had been designed separately with little focus on interaction. The disk element (Poulsen & Damkilde, 2000) was combined with bars and beams but still only in 2D. In 2007 (Niebling *et al.*, 2007) formulated a 3D beam element for reinforced concrete based on a 3x3 zone model. The zone model utilized the previously established yield surfaces for disks in the zones and thereby circumvented the formulation of the total yield surface for the beam with all of its special cases. Similarly a shell element was established on the basis of zone models, once again utilizing the disk yield surfaces. The combination of the beam element and the shell element made it possible to analyze 3D structures. For evaluation of the assumptions made in the zone models of the 3D beam element and the shell element a demand for a solid 3D modeling tool has evolved. In the area of geotechnics, solid 3D elements have been established e.g. for Mohr-Coulomb materials (Krabbenhøft *et al.*, 2008). The 3D formulation of Mohr-Coulomb materials leads to Semidefinite Programming (SDP) problems, see e.g. (Krabbenhøft *et al.*, 2008). Today only a few optimization

programs has the capability to solve SDP problems and the present work uses sedumi (Sturm & Sturm, 1999; Pólik, 2005) for this task and Yalmip (Löfberg, 2004) is used as interface to the solver.

For reinforced concrete the solid modeling in 3D of the reinforcement has to be addressed. This can be handled either by modeling of each reinforcement bar with realistic dimensions or by simply adding a tensile capacity in the direction of the reinforcement corresponding to the amount of reinforcement (Nielsen, 2008). The second choice is the basis for the solid 3D modeling of reinforced concrete in the present article. This makes geometric modeling of the structure much simpler but it also introduces some assumptions concerning the transfer of stresses between the concrete and the reinforcement. The example verifies that the material model works as intended and the results are compared with some analytical solutions.

## 2 Finite Element Formulation of the Lower Bound Theorem

The lower bound theorem of limit analysis states that a *safe and statically admissible stress distribution will not be able to cause collapse in the structure*. If we let all external load components be proportional to the load factor  $\lambda > 0$  then will all solutions in which  $\lambda \leq \lambda_p$ , where  $\lambda_p$  is the collapse load, be a lower bound solution. In practice we are not just interested in any lower bound solution i.e.  $0 \leq \lambda \leq \lambda_p$  but rather the solution closest to the collapse load i.e.  $\min(\lambda_p - \lambda)$ . This leads to an optimization problem which is often difficult, if not impossible, to solve manually.

In this paper we present a finite element framework for limit analysis of solid reinforced concrete structures. The three dimensional stress field is approximated using four node tetrahedral elements with linear stress distribution. Because of the convex properties of the yield criterion, a strict lower bound solution can be obtained by enforcing the yield criterion at the nodes.

### 2.1 Statically admissible stress distribution

As prescribed by the lower bound theorem, the chosen stress distribution must be statically admissible. For this requirement to be fulfilled when implemented in a finite element system, the internal stress state within each element as well as the inter-element stresses must be in equilibrium and the static boundary conditions satisfied.

#### 2.1.1 Internal element equilibrium

Let the three dimensional stress state at a point within the volume be defined by the stress tensor

$$\sigma_{ij} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \quad (2.1)$$

where  $\sigma_i$  are the normal stresses and  $\tau_{ij}$  for  $i \neq j$  are the shear stresses. The equilibrium equation can, with application of the symmetric convention for  $i$  and  $j$ , be written as

$$\lambda f_i + \sigma_{ji,j} = 0 \quad (2.2)$$

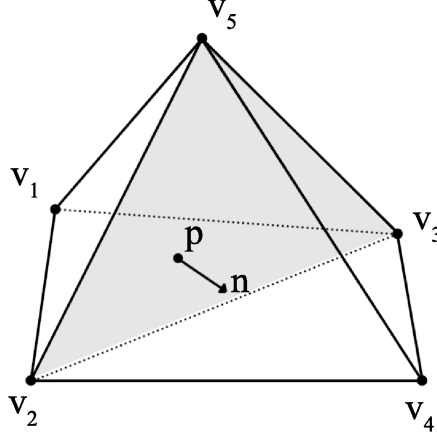


Figure 1: Inter-element equilibrium is ensured by traction continuity at the interfaces between elements.

where  $f_i$  are the proportional body forces per unit volume and  $\lambda$  is the load factor. The tensor notation  $\sigma_{ji,j}$  refers to the partial derivative of component  $ji$  with respect to direction  $j$ . For the four node tetrahedral element with a linear stress field, the partial derivatives becomes constant and the body force field is then approximated by piecewise constant body forces.

### 2.1.2 Inter-element equilibrium

Inter-element equilibrium is ensured by requiring traction continuity at the interface between adjacent elements. Figure 1 shows two tetrahedral elements which shares a common boundary defined by the vertices  $v_2, v_3$  and  $v_5$ . If  $p$  is a point on the interface and the stress state in each element at that point is defined by  $\sigma_{ij}^{p1}$  and  $\sigma_{ij}^{p2}$  respectively, the inter-element equilibrium can be written as

$$(\sigma_{ij}^{p1} - \sigma_{ij}^{p2})n_j = 0 \quad (2.3)$$

where  $n_j$  is the unit normal vector of the plane defined by  $v_2, v_3$  and  $v_5$ . As seen here, the stress state does not have to be continuous from one element to the next as long as the tractions on the shared boundary are equal.

### 2.1.3 Static boundary conditions

Surface traction equilibrium in global coordinates gives

$$-\lambda t_i + \sigma_{ij}n_j = 0 \quad (2.4)$$

where  $t_i'$  are the proportional surface tractions,  $n_j$  is the outward unit surface normal and  $\lambda$  is the load factor. If one or more traction components are supported at a point on the boundary, the associated equations are removed from the system of equations, allowing the stresses to vary freely within the yield constraints.

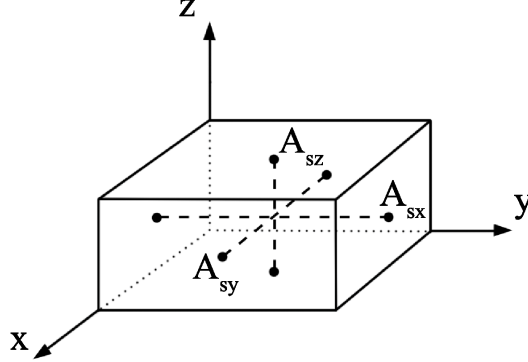


Figure 2: Solid concrete with orthogonal reinforcement.

#### 2.1.4 Safe stress distribution

A volumetric infinitesimal domain of concrete is reinforced in directions equivalent to the usual Cartesian coordinate system as shown on Figure 2. The stresses on the boundary of the infinitesimal domain are given by  $\lambda(\sigma_x^0, \sigma_y^0, \sigma_z^0, \tau_{xy}^0, \tau_{xz}^0, \tau_{yz}^0)$  where  $\lambda$  is the load factor. The shear stresses in the rebars are disregarded and the normal stresses are smoothed over the cross section area perpendicular to each of the three directions. These are denoted  $\sigma_{si}$  where  $i \in \{x, y, z\}$ . Stress equilibrium on the boundary gives

$$\begin{aligned} \lambda \sigma_x^0 &= \sigma_{sx} A_{sx} + \sigma_{cx} & \lambda \sigma_y^0 &= \sigma_{sy} A_{sy} + \sigma_{cy} & \lambda \sigma_z^0 &= \sigma_{sz} A_{sz} + \sigma_{cz} \\ \lambda \tau_{xy}^0 &= \tau_{cxy} & \lambda \tau_{xz}^0 &= \tau_{cxz} & \lambda \tau_{yz}^0 &= \tau_{cyz} \end{aligned} \quad (2.5)$$

where  $A_{sx}$ ,  $A_{sy}$  and  $A_{sz}$  are the reinforcement area per unit area perpendicular to the three axis and  $(\sigma_{cx}, \sigma_{cy}, \sigma_{cz}, \tau_{cxy}, \tau_{cxz}, \tau_{cyz})$  are the concrete stresses. If the reinforcement is not aligned with the Cartesian coordinate system, the equivalent orthotropic reinforcement values are used.

Since the shear stresses in the reinforcement are disregarded, the yield criterion for the rebar's can be formulated as simple upper- and lower bounds on the normal stress components as

$$f_{yc,i} \leq \sigma_{si} \leq f_{yt,i} \quad i \in \{x, y, z\} \quad (2.6)$$

where  $f_{yc,i}$  and  $f_{yt,i}$  are the compression and tensile strengths respectively.

## 2.2 Modified Coulomb material using Semidefinite Programming

The modified Coulomb criterion is described in (Nielsen, 1999) and is the most commonly used yield criterion for limit analysis of concrete structures. It consists of the Coulomb sliding failure criterion combined with a separation criterion. If tensile stresses are positive, the sliding failure can, in the three dimensional case, be written as

$$k\sigma_{c1} - \sigma_{c3} \leq f_c \quad (2.7)$$

where  $\sigma_{c1}$  and  $\sigma_{c3}$  are the largest and smallest principal stress respectively.  $f_c$  is the uniaxial compression strength of the concrete and  $k$  is a friction parameter. Here is used  $k = 4$  corresponding

to a friction angle  $\phi = 37^\circ$ .

The separation criterion can be formulated as

$$\sigma_{ct} \leq f_{ct} \quad (2.8)$$

where  $f_{ct}$  is the separation strength, which for a typical concrete is equal to the uniaxial tensile strength.

### 2.2.1 Semidefinite Programming (SDP)

The application of SDP in limit analysis of Coulomb materials has been described in (Krabbenhøft *et al.*, 2008), (Martin & Makrodimitopoulos, 2008) and (Bisbos & Pardalos, 2007).

Semidefinite Programming considers the problem of minimizing a linear function of the variables  $x \in \Re^m$  subjected to a set of matrix inequalities and equality constraints as

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && F(x) \succeq 0 \\ &&& Ax = b \end{aligned} \quad (2.9)$$

where the constraint function

$$F(x) = F_0 + \sum_{i=1}^m x_i F_i \quad (2.10)$$

is an affine combination of symmetric matrices. The problem data are the vectors  $c \in \Re^m$  and  $b \in \Re^k$ , the matrix  $A \in \Re^{k \times m}$  and the  $m+1$  symmetric matrices  $F_0, \dots, F_m \in \Re^{m \times m}$ . The inequality  $F(x) \succeq 0$  is called a *Linear Matrix Inequality* or *LMI* and states that the constraint function is positive semidefinite, i.e.  $x^T F(x) x \geq 0$  for all  $x \in \Re^m$ . Since positive semidefinite cones are convex, (Vandenberghe & Boyd, 1996), the SDP is a convex optimization problem.

Of course if the matrix  $F \succeq 0$  then

$$F + \lambda_F I \succeq 0 \quad (2.11)$$

for all  $\lambda_F \geq \lambda_{max}$  where  $\lambda_{max}$  is the largest eigenvalue of  $F$ .

### 2.2.2 SDP formulation of the Modified Mohr-Coulomb criterion

As described in section 2.1.4 the stress state at a point consists of a combination of concrete- and reinforcement stresses, Eq.( 2.5). The reinforcement constraints can simply be formulated as linear inequalities based on Eq. (2.6) but the modified Coulomb criterion must be cast as semidefinite constraints.

The Coulomb criterion in Eq. (2.7) can be written as a combination of two positive semidefinite cones, see (Krabbenhøft *et al.*, 2008; Martin & Makrodimitopoulos, 2008; Bisbos & Pardalos, 2007). Eq. (2.12) and (2.13) shows this formulation using the concrete material parameters  $f_c$  and  $k$  as posed by (Nielsen, 2008).

$$\sigma_{ij}^c + k\alpha I \succeq 0 \quad (2.12)$$

$$-\sigma_{ij}^c + \left( \frac{f_c}{k} - \alpha \right) I \succeq 0 \quad (2.13)$$

where  $\alpha$  is an auxiliary variable,  $\mathbf{I}$  is the unit matrix and  $\sigma_{ij}^c$  is a stress tensor containing the concrete stresses.

The separation criterion given in Eq. (2.8) can be formulated directly as an LMI constraint on the form

$$-\sigma_{ij}^c + \mathbf{I} f_{ct} \succeq 0 \quad (2.14)$$

### 2.3 Solving lower bound problems using the YALMIP interface

Implementing the linear constraints posed by the equilibrium constraints and the yield criterions for the reinforcement is straight forward since they are part of the general formulation of the SDP problem, Eq. (2.9). Most SDP solvers only allow for a single LMI constraint on a set of variables, so the modified Coulomb criterion described in section 2.2.2 proves a bit more difficult to implement. Instead of reformulating the problem to fit the format of a specific solver, it is chosen to use the YALMIP interface, (Löfberg, 2004). YALMIP provides a parametric modeling language that enables developers a high-level model for defining different types of optimization problems. One of the main advantages of using YALMIP is that it takes care of all the low-level modeling and is designed to obtain as efficient and numerical sound model as possible. Another advantage is that it supports a wide range of solvers, making it possible to switch from one solver to another by simply changing the optimizer settings. YALMIP is implemented as a MatLab<sup>TM</sup> toolbox.

## 3 Numerical Examples

This section shows how the numerical lower bound model presented here can be used to solve some basic examples. There are currently only a few solvers capable of solving SDP problems. Here we use the open source solver SeDuMi, (Sturm & Sturm, 1999; Pólik, 2005) which is implemented as a MatLab<sup>TM</sup> toolbox <sup>1</sup>. All default settings are used except for the way SeDuMi handles free variables. By default, SeDuMi places free variables in a quadratic cone which caused stability problems. These problems were resolved by forcing SeDuMi to split the free variables instead of placing them in cones.

All tests are performed on a Mac Pro Workstation (2 x 2.8 GHz Quad Core Xeon, 6 GB RAM) running Windows XP Pro x64 under Bootcamp.

### 3.1 Numerical determination of yield condition for a reinforced solid

In the first example, the numerical model will be used to determine the yield conditions for a solid block of reinforced concrete with various types of reinforcement. For these tests, a cubic block with a side length of 5 is modeled using a structured mesh of  $3 \cdot 3 \cdot 3 \cdot 24 = 648$  elements as shown on Figure 3. The yield surface is a function of the six independent stress parameters in the stress tensor  $\sigma_{ij}$ . To simplify visualization, only a section in the  $\sigma_x \tau_{xy}$ -plane is determined here. This is

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<sup>1</sup>The 64 bit version of SeDuMi bundled with CVX is used (Grant & Boyd, 2008a; Grant & Boyd, 2008b).

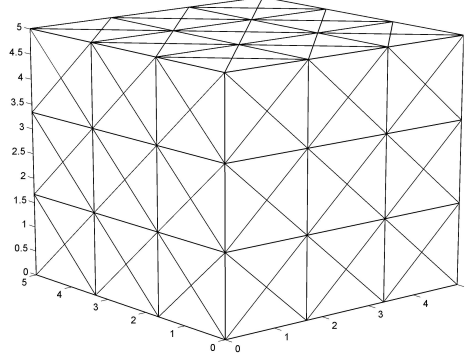


Figure 3: Square block of reinforced concrete discretized using a structured mesh.

done by applying cosine- and sine weighted values of  $\sigma_x$  and  $\tau_{xy}$  to the block

$$\tau_{xy}^0 = \lambda \sin(v) \quad \sigma_x^0 = \lambda \cos(v) \quad (3.1)$$

were  $0 \leq v \leq \pi$ .

The following material parameters are used unless otherwise noted

$$\begin{aligned} f_c &= 1 & f_{ct} &= 0.0 & k &= 4 \\ f_{yt} &= 1 & f_{yc} &= 1 & & \text{in the } x,y,z \text{ directions} \end{aligned}$$

### 3.1.1 Isotropic Disc

In (Nielsen, 1999) the yield condition for a reinforced concrete disc is determined based on the lower bound theorem. In this example we will show how the yield surface for a disk can be replicated using the numerical model. The complete surface is a bit complex, but figure 2.2.11 in (Nielsen, 1999) shows the curve of intersection between the yield surface and the  $\sigma_x \tau_{xy}$ -plane, and the expression for this curve is summarized below

$$\tau_{xy}(\sigma_x) = \begin{cases} \sqrt{\Phi f_c (\Phi f_c - \sigma_x)} & \text{if } -(1 - 2\Phi)f_c \leq \sigma_x \leq \Phi f_c \\ \sqrt{\Phi (1 - \Phi) f_c} & \text{if } -f_c \leq \sigma_x \leq -(1 - 2\Phi)f_c \\ \sqrt{\frac{1}{4}f_c^2 - [\sigma_x + (\frac{1}{2} + \Phi)f_c]^2} & \text{if } -(1 + \Phi)f_c \leq \sigma_x \leq -f_c \end{cases}$$

The isotropic disc is easily modeled by simply removing the reinforcement in the direction perpendicular to the loaded plane, in this case the z-direction

$$\begin{aligned} \Phi_x &= \Phi_y = 0.1 \\ \Phi_z &= 0.0 \end{aligned}$$

Figure 4 shows the  $\tau_{xy} \sigma_x$ -section of the yield surface for both the analytical and the numerical model. As seen from the figure, a very good correlation between the analytical and the numerical model is achieved. The difference observed on the figure is caused by the subdivision with which the numerical curve is generated.



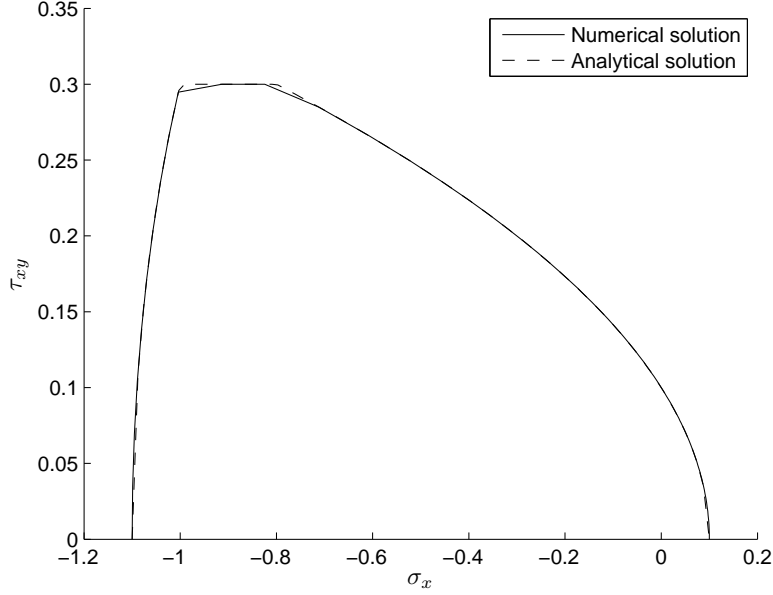


Figure 4: Comparison between the numerical and the analytical yield surface for a reinforced concrete disk. The numerical curve is determined using 100 linearly spaced values of  $v$  in the interval 0 to  $\pi$ .

### 3.1.2 Isotropic Solid

The previous example showed how the numerical model could be used to recreate a known yield condition for an isotropic disc. This example investigates the yield conditions for an isotropic solid of reinforced concrete, i.e.

$$\Phi_x = \Phi_y = \Phi_z = 0.1$$

Figure 5 shows the  $\tau_{xy}$ - $\sigma_x$ -section of the yield surface as determined by the numerical analysis and compared with the disc solution, some tri-axial effects are seen. One thing to note is the effects on the uniaxial compression strength, which here is determined to 1.5. This is identical to the theoretical value which can be found by assuming full utilization of the transverse reinforcement

$$\sigma_{sy} = \sigma_{sz} = A_s f_y \quad (3.2)$$

The concrete stress components in the y- and z-direction can be determined from Eq. (2.5) when  $\sigma_y^0 = \sigma_z^0 = 0$  as

$$\sigma_{cy} = \sigma_{cz} = -A_s f_y \quad (3.3)$$

With no shear stresses acting on the cube, the normal stresses are equal to the principal stresses (no off-diagonal elements in the concrete stress tensor). Since the uniaxial compression strength is greater than the transverse compression generated by the reinforcement, the principal stresses in the concrete material becomes

$$\sigma_{c1} = \sigma_{c2} = -A_s f_y \quad \sigma_{c3} = \sigma_x^0 - (-f_y)A_s \quad (3.4)$$

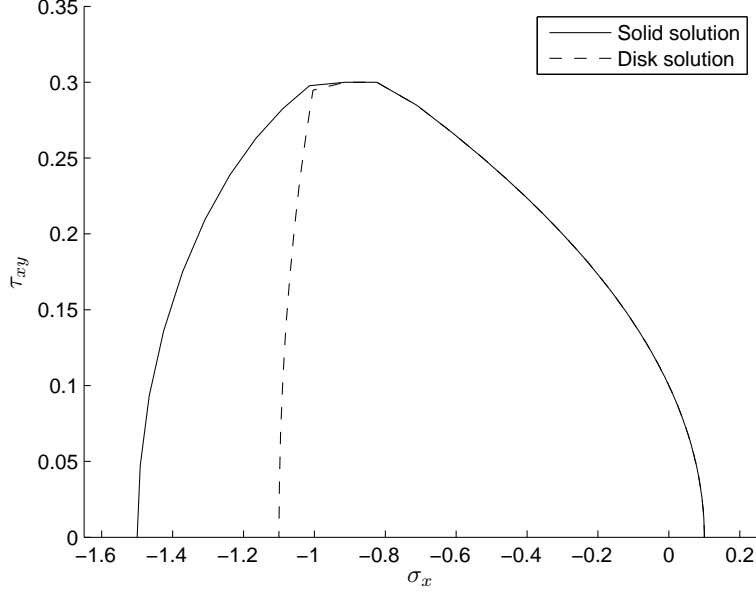


Figure 5: The yield surface of a reinforced concrete solid determined using the numerical model. The numerical curve is determined using 100 linearly spaced values of  $v$  in the interval 0 to  $\pi$ .

The uniaxial compression strength can be determined from the sliding criterion given in Eq. (2.7).

$$\sigma_x^0 \leq f_c(1 + \Phi(k + 1)) = 1.5 \cdot f_c \quad (3.5)$$

## 4 Conclusions

We have shown a method for performing limit analysis of reinforced concrete structures using a general finite element framework based on lower bound elements. The method uses Semidefinite Programming algorithms for solving the optimization problem posed by the lower bound method. The solver used here (SeDuMi) has proven very efficient and robust for these types of problems and the use of YALMIP as interface made it very easy to formulate the optimization problems. To simplify modeling of complex structures a material model in which reinforcement and concrete are considered as a homogeneous material. The stress state in this material is defined by a combination of concrete- and reinforcement stresses and yield criteria are imposed on these stress components separately. The numerical model has been verified by comparison with well known analytical solutions and good correlation has been found between the results. The approximation of considering the reinforcement and concrete as a unified material has also been tested and the results are very encouraging.

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